Time series analysis seminar

2019/2020 spring semester

Game rules

- You can miss at most 3 seminars without consequences. If you are absent from more, you will not receive a grade.
- 100 + x points are achievable during the semester:
  - · 32 points: small homework (4p · 8) – exercises with the sign **B**
  - · 28 points: big homework
  - · 40 points: classroom test: V.12., D 0-803 Szabó József előadó
  - · x points: additional homework – exercises with the sign **SZ**
- On the classroom test you have to achieve at least 30%. If you fail the test, you do not participate on it or you would like to rewrite it, then you will have opportunity to do a make-up test on the first week of the exam period. If you rewrite the test, the better result will be taken into account. If you fail the make-up test you have to write a make-up practice exam, the points for the homework remain. In case of writing a make-up practice exam, you cannot get a better mark than a 2.
- On the test you are allowed to use: **calculator** and a cheat-sheet written by YOUR OWN HANDWRITING on an at most A4 sized paper.
- Small homework: Each of them for maximum 4 points, the best 8 of them will be taken into account. There will be approximately 12 small homework, from which you can choose to solve. The goal of the small homework is to make you learn, practice during the whole semester, so you will be able to hand them in until a fix deadline.
- Big homework: until a fix deadline, a bigger essay-like analysis
- Additional homework: until the second test you can hand in whenever and whichever you intend
- When working on your homework, it is absolutely NOT forbidden to consult with you classmates – but I expect you to **write down your own thoughts and conclusions**. If I spot obvious signs of copying in a homework, then every student involved gets 0 points.
- Grading:

1	0	–	34,99
2	35	–	49,99
3	50	–	64,99
4	65	–	79,99
5	80	–	∞^3

Information about the lecturer

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Recommended literature

- Shumway-Stoffer: Time series analysis and its applications, with R examples Access on the web: `https://www.stat.pitt.edu/stoffer/tsa4/tsa4.pdf`
- Brockwell-Davis: Introduction to time series and forecasting

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Software/programming language used on the seminar: **R**

- One of the best programming languages for statistical modelling and data analysis (next to Python and Matlab)
- Open source, there is a package in **R** for almost every statistical problem to solve it
- The most widely used programming language for math purposes.
- Download: `https://cran.r-project.org/`
- Recommended software: RStudio; download: `https://www.rstudio.com/products/rstudio/download3/`

If an exercise does not mention it, then t takes its value on the set of integer numbers, i.e. $t \in \mathbb{Z}$.

1.) Let X_t be a *random walk with drift*: $X_t = \delta + X_{t-1} + \varepsilon_t \quad (t = 1, 2, \ldots)$, where δ is a real parameter, $P(X_0 = 0) = 1$ és $\varepsilon_t \sim WN(0, \sigma^2)$.

a.) Express X_t in terms of the white noise!

b.) Let us calculate the expectation, autocovariance and autocorrelation functions of X_t! Is the process weakly stationary?

c.) Let us simulate from such a process, if the innovations are Gaussian, $\delta = -1, 0, 1$ and $\sigma = 1, 5, 10$! Let us depict X_t, the trend and the autocorrelation function!

d.) Does there exist a $\varphi(X_t, X_{t-1}, \ldots)$ transformation that makes the process stationary?

e.) Let us assume that the white noise ε_t is Gaussian. Determine the ML (maximum likelihood) estimate of the parameters δ and σ^2 using the first n observations as a sample of n elements! Is the estimate of δ unbiased/consistent?

2.) Let $X_t = \frac{1}{4}\varepsilon_{t-1} + \frac{1}{2}\varepsilon_t + \frac{1}{4}\varepsilon_{t+1} \quad (t = 1, 2, \ldots)$, where ε_t is a white noise.

a.) Determine the expectation, autocovariance and autocorrelation functions of X_t! Is the process weakly stationary?

b.) Simulate a white noise, create the process X_t, then make a picture consisting the 2 graphs together! What can we see? Let us interpret the results!

3.) Let $X_t = \alpha + \beta t + \varepsilon_t \quad (t = 1, 2, \ldots)$, where α, β are real parameters and ε_t is a white noise.

a.) Determine the expectation, autocovariance and autocorrelation functions of X_t! Is the process weakly stationary?

b.) Does there exist a $\varphi(X_t, X_{t-1}, \ldots)$ transformation that makes the process stationary?

4.) Let $(X_t)_{t\in\mathbb{R}} = U\sin(2\pi\alpha t) + V\cos(2\pi\alpha t)$, where α is a real parameter, U and V are independent random variables with 0 expected value and τ standard variance.

a.) Determine the expectation, autocovariance and autocorrelation functions of X_t! Is the process weakly stationary?

b.) Let $Y_t = X_t + \varepsilon_t$, where ε_t is white noise. Simulate a white noise with $D\varepsilon_t = 1, 2, 4, 8$; then create Y_t-t, if $\tau = 2$ and $\alpha = 1$; at last, depict the time series X_t and Y_t! Let us interpret the results!

5.) Let $X_t = \sin(2\pi U t)$ $(t = 1, 2, \ldots)$, where $U \sim E(0; 1)$. Show that X_t is weakly stationary, but it is *not* strictly stationary!
Hint for the strict stationarity: compute the following probabilities: $P\left(X_1 < -\frac{\sqrt{3}}{2}, X_2 < -\frac{\sqrt{3}}{2}\right)$ és $P\left(X_2 < -\frac{\sqrt{3}}{2}, X_3 < -\frac{\sqrt{3}}{2}\right)$!

B1.) **[II.25.]** Let us consider the following $(X_t)_{t=1,2,\ldots}$ process: $X_t = ae^{X_{t-1}} + \varepsilon_t$, where a is a real parameter, ε_t is a Gaussian white noise with 0 expected value and unknown σ standard deviation, $X_0 = x_0 \in \mathbb{R}$. Deduce the ML estimate of the unknown a parameter!

B2.) **[II.25.]** Let us take the following $(X_t)_{t\in\mathbb{Z}}$ process: $X_t = \varepsilon_t\eta_t$, where ε_t and η_t are independent processes (from each other and in time as well), ε_t is a white noise with 0 expectation and σ standard deviation, $P(\eta_t = 0) = P(\eta_t = 2) = \frac{1}{2}\ \forall t$. Is the process X_t weakly stationary? Compute the autocovariance and autocorrelation functions!

B3.) **[III.3.]** Take a thorough look at the **R** introduction on my homepage and solve the following problem with **R**! Send me the source code of your solution to my Email address!
We sample 4 cards with replacement from a traditional full Hungarian card deck. Calculate with simulation the probability of choosing 2 leaves (zöld/green)! At least how many repetitions would you recommend so that the simulated value approximates the real probability with an error at most 0.5%? State your answer with an appropriate figure showing the rate of convergence!
Just for curiosity, a quite recent Hungarian advertisement with the leaves ace.

B4.) **[III.3.]** Take a thorough look at the **R** introduction on my homepage and solve the following problem with **R**! Send me the source code of your solution to my Email address!
Simulate 10000 pieces of exponential samples of sizes 1000 with parameter $\lambda = 2$! Determine the ML estimate for each sample (notation: λ_{ML})! We have learnt in introductory statistics that under some regularity conditions the ML estimate is asymptotically Gaussian. Having this result in mind, look at the ML estimates as a sample from a normal distribution and estimate the unknown parameters of this distribution! Create a histogram with 50 breakpoints from the λ_{ML} values, and depict the histogram together with the density function of the normal distribution!

SZ1.) Let $(X_t)_{t\in\mathbb{R}}$ be a stationary Gaussian process (i.e. for each $k \in \mathbb{Z}_+$ and $t_1, \ldots, t_k \in \mathbb{R}$, the random vector $(X_{t_1}, \ldots, X_{t_k})$ is jointly Gaussian distributed) and $Y_t = e^{X_t}$.

a.) Determine the moment generating function $M_{X_t}(s) = E(e^{sX_t})$ of process X_t!

b.) Show that process Y_t is weakly stationary! (1+1= 2p)

SZ2.) Let $X_t = at + W_t$ $(0 \le t \in \mathbb{R})$, where a is an unknown real parameter, W_t is a Wiener process. Determine the ML estimate of parameter a, if there is a sample of n elements at hand for equal time steps, i.e. let $\delta > 0$ be fixed, $t_i - t_{i-1} = \delta$ $(i = 1, 2, \ldots, n)$, $t_0 = 0$; and the empirical sample is $x_{t_1}, x_{t_2}, \ldots, x_{t_n}$ – we "look" at the process at regular time intervals. (2p)
Hint: It is easy to see that $X_{t_j} = X_{t_{j-1}} + a\delta + W_{t_j} - W_{t_{j-1}}$ $(j = 1, 2, \ldots, n)$, which makes it a piece of cake to determine the conditional distribution of $X_{t_j}|X_{t_{j-1}}$.

6.) Let us assume that ε_t is an i.i.d. white noise with 0 expectation and $\sigma_\varepsilon = 1$ standard deviation. Let us determine whether the following processes are causal (have a stationary $MA(\infty)$ representation) and whether they are invertible (they have an $AR(\infty)$ representation)!

a.) $X_t = 2 + 0,9X_{t-1} + \varepsilon_t$

b.) $X_t = 0,2X_{t-1} + 0,35X_{t-2} + \varepsilon_t$

c.) $X_t = 0,5X_{t-1} - 0,25X_{t-2} + \varepsilon_t$

d.) $X_t = X_{t-1} - 0,25X_{t-2} + \varepsilon_t$

e.) $X_t = \varepsilon_t + 0,5\varepsilon_{t-1}$

f.) $X_t = 0,5X_{t-1} + \varepsilon_t - 0,5\varepsilon_{t-1}$.

g.) $X_t = 0,9X_{t-1} + \varepsilon_t + 0,5\varepsilon_{t-1}$.

Let us determine the followings:

- expectation and standard deviation of the stationary solution, autocorrelation function;
- $MA(\infty)$ representation of the process;
- spectral density.

Let us simulate such a process with **R**! Let us determine with the help of **R** functions the autocorrelation, partial autocorrelation functions, the elements of the $MA(\infty)$ representation and the spectral density function!

7.) Let us generate a sample of 1000 elements from $X_t = \omega + \alpha \cdot X_{t-1} + \varepsilon_t$ process, where $\omega = 2$, $\alpha = 0,6$ és $\varepsilon_t \sim WN(0; 1)$. Let us estimate the parameters ω and α

- with function `arima` in case of different methods (ML estimation, least square estimation);
- with function `lm`, i.e. looking at the problem as linear regression.

Let us compare the estimates and interpret the results! Let us try to find an explanation to the odd results!

8.) Let $X_t = \alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \varepsilon_t$, where ε_t is i.i.d. noise with finite variance. Prove that the process has a stationary solution if and only if $\alpha_1 + \alpha_2 < 1$, $\alpha_2 - \alpha_1 < 1$ and $|\alpha_2| < 1$!

9.) Generate a sample of size $n = 200$ from a zero-mean AR(2) process with parameters $\alpha_1 = 0,5$ és $\alpha_2 = 0,3$.

a.) Estimate back the parameters and do model diagnostic testing – investigate how good the estimates and the model are!

b.) Play a bit with fitting other models (for instance AR(1), AR(3), ARMA(1,1)). How do we spot that these models are inappropriate?

c.) Investigate via simulations how the AIC and BIC information criteria perform in choosing the appropriate model order! Let us compare the AIC/BIC values, if we fit AR(1), AR(3), AR(4), ARMA(1,1) models to sample from the AR(2) process! What happens, if the sample size increases?

10.) Let us analyse and fit a proper ARMA model to the time series that can be found in file 'gnp96.txt', which contains the real GNP of the USA from 1947 until 2002. These data are quarterly, calculated to 1996 prices, the seasonal components have already been eliminated.

a.) Let us depict the time series and look at the empirical ACF/PACF functions! Does the time series look stationary?

b.) Applying a proper transformation (taking logdifferences and/or fitting a regression model), let us try to make it stationary!

c.) After analysing the empirical ACF/PACF function let us choose a proper ARMA model with AIC/BIC criteria! Let us do model diagnostics (are the estimated parameters significant, can the residuals stem from a white noise process)!

11.) Similarly to the previous problem let us analyse and fit a proper ARMA model to the 'gtemp' time series found in the 'astsa' package that contains the average temperature of the Earth between 1880 and 2009.

12.) Let us examine the difference-sign test! Let $X_1, \dots, X_n$ sample from a process, and let Z_n be the number of time point where the value of the process increases, i.e. $Z_n = \sum_{i=2}^{n} Y_i$, where $Y_i = I(X_i > X_{i-1})$, $i = 2, 3, \dots, n$.

a.) Using the central limit theorem of dependent random variables (practice notes, page 3) show that $\frac{Z_n - \frac{n-1}{2}}{\sqrt{\frac{n-1}{12}}} \xrightarrow[n\to\infty]{d} N(0;1)$!

b.) Generate i.i.d. series with sample sizes $n = 10, 50, 100, 200, 500, 1000, 2000$, then estimate the probability of the type I error with the help of simulations!

c.) Generate samples of sizes $n = 10, 50, 100, 200, 500, 1000, 2000$ from

- AR(1) processes;
- random walks with drift,

then estimate the power difference-sign test!

B5.) **[III.10]** Let us consider the process $X_t = \alpha X_{t-1} + \alpha^2 X_{t-2} + \varepsilon_t + \beta\varepsilon_{t-1}$, where $\varepsilon_t \sim WN(0, 2^2)$. For which values of parameters $\alpha \in (0,1)$ and $\beta \in (0,1)$ will X_t be a causal ARMA(2,1) process? For which parameter values will it be a causal AR(1) process?

B6.) **[III.31]** Let us take the following empirical estimates from a sample of size 1000:

Lag	0	1	2	3	4	5	6	7
Empirical autocovariance	0.620	-0.200	0.005	-0.013	0.005	-0.012	-0.028	0.017
Emp. partial autocorr.	1	-0.323	-0.107	-0.060	-0.021	-0.028	-0.070	-0.017

Which model would you choose from the ARMA family to model the sample? Estimate the unknown parameters (using the theoretical values of the autocorrelation function)!

B7.) **[V.12.]** Let $X_t = 0,3X_{t-1} + 0,04X_{t-2} + \varepsilon_t$, where $\varepsilon_t \sim WN(0, \sigma_\varepsilon)$.

a.) Is the process stationary/causal?

b.) Is the process invertible?

c.) Give the $MA(\infty)$ representation of the process! (1+1+2= 4p)

SZ3.) Let $X_t = 0,4X_{t-1} + 0,45X_{t-2} + \varepsilon_t + \varepsilon_{t-1} + 0,25\varepsilon_{t-2}$, where ε_t is a white noise. Does the process have a stationary solution? If yes, give the MA(∞) representation! (2p)

SZ4.) Show that the partial autocorrelation of the MA(1) process $X_t = \varepsilon_t + \beta\varepsilon_{t-1}$ is $\rho_k = \frac{-(-\beta)^k}{1+\beta^2+\ldots+\beta^{2k}}$! (2p)

SZ5.) Let $X_1, \dots, X_n$ be a sample from a white noise. Show that

a.) the matrix $\mathbf{W}$ in the Barlett theorem (see practice notes) is a identity matrix.

b.) Construct confidence interval to the emprical autocorrelation $\hat{r}(h)$, $h \in \mathbb{Z}$! (1+1= 2p)

SZ6.) Prove that if the sample stems from an i.i.d. white noise (the null hypothesis holds), then the test statistic of the turning point test converges to the standard normal distribution! (2p)

13.) Let X_t $(t = 1, 2, \ldots)$ be an ARCH(1) process, i.e. $X_t = \sigma_t\varepsilon_t$, where $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2$ and ε_t is a Gaussian white noise.

a.) Determine the $X_t|\mathcal{F}_{t-1}$ conditional distribution, where $\mathcal{F}_t = \sigma(X_s : s \le t)$!

b.) Calculate the X_t expected value!

c.) Calculate $\mathrm{cov}(X_{t+h}, X_t)$, if $h > 0$!

d.) Determine the standard deviation and the 4th moment of X_t! When do these values exist?

e.) Show that X_t^2 is an AR(1) process!

Let us simulate ARCH(1) processes with **R** and calculate the values above!

14.) Let us fit an appropriate GARCH model to the

a.) NYSE;

b.) GBP-CHF;

time series! Does the model fit well?

SZ7.) In case of a stationary ARCH(1) process, calculate the kurtosis$= \frac{E(X_t - EX_t)^4}{D^4 X_t} - 3$ value. For which parameter values is the process leptokurtic, i.e. the kurtosis of the process is larger than the kurtosis of standard Gaussian distribution? (1p)

15.) Let $(X_n)_{n\in\mathbb{Z}}$ be a stationary process with μ expectation and $R(h)$ autocovariance function. Let us find the $0 < h \in \mathbb{Z}$ step least square linear prediction with the help of X_n, i.e. determine the a and b values, for which $MSE(a,b) := E([X_{n+h} - (a + bX_n)]^2)$ is minimal! Let us calculate the standard deviation of the estimate $\hat{X}_{n+h} = \hat{a} + \hat{b}X_n$!

16.) X_t is a stationary AR(1) process with 0 expectation and the standard deviation of the innovations is σ.

a.) Determine the 1-step, then the h-step least square forecast of the process and calculate the error of the forecast! Compare the results with the results of Exercise 15 !

b.) Let us assume that we know the 1st and the 3rd observation, but the 2nd observation is missing. Give least square linear estimation for X_2 using X_1 and X_3, and calculate the error of the forecast!

17.) Let X_t be a stationary MA(1) process with 0 expectation, the standard deviation of the innovations is σ. We know the 1st and the 2nd observation and want to estimate the 3rd observation. Give least square linear estimation for X_3 using X_1 and X_2, and calculate the error of the forecast!

18.) Let X_t be a stationary AR(p) process with 0 expectation, the standard deviation of the innovations is σ. Determine the 1-step least square forecast of the process!

19.) Let X_t be a stationary AR(1) process with μ expectation, the standard deviation of the innovations is σ. Using the properties of the forecast operator, determine the h-step least square forecast of the process!

20.) Determine the 10-step best least square linear forecast for the time series found in file 'gnp96.txt', using the best ARMA model, with the help of the built-in function `predict`! Give interval estimation to the forecasts! Depict the time series with the forecasts!

B8.) **[IV.21.]** Let $X_1, X_2, \ldots, X_n$ be a sample from the process $X_t = -\pi + \varepsilon_t + 2\varepsilon_{t-1} - \varepsilon_{t-3}$, where ε_t is a white noise with σ standard deviation.

a.) Determine the h-step least square linear forecast using X_1 and X_2 values for each $h = 1, 2, \ldots$! Calculate the error of the forecast!

b.) Let us assume that $n = 1000$. Give a least square estimation for value X_{100}, using the first two and the last two values of the sample! (2,5+1,5= 4p)

SZ8.) Let X_t be a stationary AR(1) process with α parameter, the standard deviation of the innovations is 1. Show that $D^2(\overline{X}_n) = \frac{\sigma^2}{n^2(1-\alpha^2)} \frac{n - n\alpha^2 - 2\alpha + 2\alpha^{n+1}}{(1-\alpha)^2}$. (1p)

SZ9.) Let X_t be a stationary MA(1) process with 0 expectation, the standard deviation of the innovations is σ. Let us assume that we know the 1st, 2nd, 4th and 5th observations but do not know the 3rd one. Give least square linear estimation for value X_3, using values X_1, X_2, X_4 and X_5, and determine the error of the estimation! (1p)

21.) Let us consider the $X_t = \alpha + \beta t + \varepsilon_t$ $(t = 1, 2, \ldots)$ process, where α, β are real parameters and ε_t is a white noise. What type of ARIMA process is this?

22.) Let us take the $X_t = \omega + \alpha X_{t-1} + \varepsilon_t$ AR(1) process, where $\varepsilon_t \sim WN(0, \sigma^2)$. What kind of process do we get, if we take the difference of the process once, 2 times, 3 times, etc.?

23.) Let us consider the following ARIMA(1,1,0) model: $(1 - \phi B)(1 - B)X_t = \varepsilon_t$, where $\varepsilon_t \sim WN(0, \sigma^2)$, $|\phi| < 1$, $DX_0 < \infty$, $t = 1, 2, \ldots$

a.) Let us solve the ARIMA equation, i.e. let us express X_t in terms of the noise!

b.) Let us simulate from such a process and let us look at the autocorrelation and partial autocorrelation functions! Let us estimate the parameters and conduct a unit root test as well!

24.) Using the least square estimate of the parameters of the generalised linear model, show that in case of the $y_i = ax_i + b$, $\quad i = 1, 2, \ldots, n$ univariate linear regression model the least square estimate of the coefficients is

$$\hat{a} = \frac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x})^2} = \frac{\left(\sum_{i=1}^{n} x_i y_i\right) - n \cdot \overline{x} \cdot \overline{y}}{\left(\sum_{i=1}^{n} x_i^2\right) - n \cdot \overline{x}^2} \quad \text{és} \quad \hat{b} = \overline{y} - \hat{a}\overline{x}.$$

25.) Quarterly turnover of an ice cream seller (thousand scoops):

Év	Quarter I.	Quarter II.	Quarter III.	Quarter IV.
2017	95	152	255	118
2018	102	146	248	124
2019	97	156	245	122

The elements of the time series are denoted by $(x_t)_{t=1,2,\ldots,12}$. Some calculation results: $\sum_{t=1}^{12} x_t = 1860$ $\quad \sum_{t=1}^{12} t \cdot x_t = 12342$ $\quad \sum_{t=1}^{12} t = 78$ $\quad \sum_{t=1}^{12} t^2 = 650$

a.) Determine the equation of the linear trend and interpret the parameters!

b.) Determine and interpret the seasonal indices!

c.) Give point and interval estimation for the 2020 sales volume (for all 4 quarters)!

26.) Let us eliminate the trend and seasonal components from the *jj* Johnson&Jonhson quarterly time series (1960-80) found in the *astsa* package by *jj*

a.) calculating seasonal indices;

b.) using function *stl* of the **R** language (seasonal decomposition of time series by LOESS regression);

c.) taking proper differencing;

d.) fitting a SARIMA model.

Let us fit an ARMA model to the residuals (if has sense), finally give forecasts for every quarter of year 1981!

27.) Let Y_t be a stationary process with 0 expectation and be a and b real constants. If $X_t = a + bt + s_t + Y_t$, where s_t is a seasonal component with period 4, then let us show that process $\nabla\nabla_4 X_t = (1 - B)(1 - B^4)X_t$ is stationary and let us express its autocovariance function with the autocovariance function $R_Y(h)$ of process Y_t!

28.) Let us consider the linear filter with coefficients $a_i = \frac{1}{1+2q}$, if $-q \le i \le q$ and

$a_i = 0$, if $|i| > q$.

a.) Let us show that this filter passes the linear trend az $m_t = c_0 + c_1 t$ without distortion.

b.) If Z_t, $t \in \mathbb{Z}$ are independent random variables with 0 expected value and σ std, then let us calculate the expected value and std of the linear filtered $U_t = \frac{1}{2q+1} \sum_{i=-q}^{q} Z_{t-i}$ process! What can we conclude in case of large q?

29.) Show that the linear filter with $(a_{-2}, a_{-1}, a_0, a_1, a_2)^T = \frac{1}{9}(-1, 4, 3, 4, -1)^T$ coefficients passes each arbitrary polynomial of order 3 and eliminates the seasonal components with period 3!

B9.) **[IV.28.]** The turnover of a scone seller at the Siófok Golden Coast (income, Thousand Forints) for different months:

Year	May	June	July	August	September
2016	513	1052	2555	2118	634
2017	631	1246	2848	2249	831
2018	459	968	2648	2224	348
2019	642	1157	2245	1822	730

The elements of the time series are denoted by $(x_t)_{t=1,2,\ldots,12}$. Some calculation results: $\sum_{t=1}^{20} x_t = 27920$ $\sum_{t=1}^{20} t \cdot x_t = 292781$ $\sum_{t=1}^{20} t = 210$ $\sum_{t=1}^{20} t^2 = 2870$

a.) Determine the equation of the linear trend and interpret the parameters!

b.) Determine and interpret the seasonal indices!

c.) Give point and interval estimation for the 2020 sales volume (for all 5 months)!

B10.) **[IV.28]** Let $Y_t = -6 + \varepsilon_t - 6\varepsilon_{t-1}$, where ε_t is a white noise with 0 expectation and 2 standard deviation, furthermore be a and b real constants. If $X_t = (a + bt)s_t + Y_t$, where s_t is the seasonal component with period 12, then show that process $\nabla_{12}^2 X_t$ is stationary and determine its autocovariance function!

B11.) **[V.5]** Determine the linear filter of form $1 + \alpha B + \beta B^2 + \gamma B^3$ – i.e. find the α, β, γ real coefficients –, which passes every linear trend without distortion and eliminates every seasonal component with 2 periods!

SZ10.) Let us consider the ARIMA model of exercise 23. Calculate the autocorrelation function! (2p)

SZ11.) Let $m_t = \sum_{i=0}^{p} c_i t^i$ be a polynomial trend, where $0 \neq c_i \in \mathbb{R}$. Calculate the $\nabla^p m_t$ and $\nabla^{p+1} m_t$ values! What do you conclude? (1p)

SZ12.) Prove that the linear filter with parameters $\{a_i\}_{i \in \mathbb{Z}}$ passes an arbitrary polynomial of degree k without distortion if and only if $\sum_{i=-\infty}^{\infty} a_i = 1$ and $\sum_{i=-\infty}^{\infty} i^r a_i = 0$ for each $r = 1, 2, \ldots, k$! (2p)

SZ13.) Construct a linear filter that passes every polynomial trend of degree 2 without distortion and eliminates every seasonal component with 4 periods! (2p)

30.) Let X_t and Y_t be with each other uncorrelated stationary processes with R_X and R_Y autocovariance functions; and spectral distribution functions F_X and F_Y. Show that for $Z_t = X_t + Y_t$ process $R_Z(h) = R_X(h) + R_Y(h)$, furthermore $F_Z(x) = F_X(x) + F_Y(x)$ holds for all $h \in \mathbb{Z}$ and $x \in \mathbb{R}$.

31.) Determine the spectral density function of the following processes and the spectral distribution function, if there does not exist the spectral density:

a.) white noise with standard deviation σ;

b.) AR(1) process: $X_t = \alpha X_{t-1} + \varepsilon_t$, where $1 > |\alpha| \in \mathbb{R}$, $\varepsilon_t \sim WN(0, \sigma^2)$;

c.) MA(1) process: $X_t = \varepsilon_t + \alpha\varepsilon_{t-1}$, where $\alpha \in \mathbb{R}$, $\varepsilon_t \sim WN(0, \sigma^2)$;

d.) $X_t = U \sin(\alpha t) + V \cos(\alpha t)$, where α is a real parameter, U and V are independent random variables with 0 expectation and τ standard deviation;

e.) $X_t = S_t + \alpha S_{t-D} + N_t$, where S_t and N_t are stationary and from each other independent processes with 0 expectation and f_S, f_N spectral densities; D is a known positive real number; α is an unknown real parameter.

Let us simulate from the processes with different parameter values, depict the empirical sample, the autocorrelation function and the spectral density function (distribution function)!

32.) Let $X_t = U \sin\left(\frac{\pi}{3}t\right) + V \cos\left(\frac{\pi}{3}t\right)$, where U and V independent random variables with 0 expected value and τ standard deviation; $Y_t = \varepsilon_t + 2,5\varepsilon_{t-1}$, where $\varepsilon_t \sim WN(0, \sigma^2)$; furthermore let $Z_t = X_t + Y_t$. Determine the autocovariance and spectral distribution functions of process Z_t!

33.) Let us decide whether the following functions defined on integer numbers can be the autocovariance function of a stationary process:

a.) $R(h) = I(h = 0)$;

b.) $R(h) = I(h = 0) - 0,5 \cdot I(|h| = 2) - 0,25 \cdot I(|h| = 3)$;

34.) Let the spectral density function of a stationary process be $f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, $x \in [-\pi; \pi]$. Determine the autocovariance and autocorrelation functions of the process!
Hint: integrate the derivative of the autocovariance function by parts, then solve the resulting $R'(h) = -hR(h)$ differential equation!

B12.) **[V.12.]** Let $X_t = \sum_{i=1}^{N} a_i \cos(\omega_i t + U_i)$, where $\omega_i \neq 0$ and a_i are real constants for all $i = 1, 2, \ldots, N$; furthermore $U_i \sim E(0; 2\pi)$ are independent. Determine the spectral density and distribution function of process X_t!

SZ14.) Determine the values of parameter c for which the function $R(h) = I(h = 0) + cI(|h| = 1)$ with integer domain may be an autocovariance function of a stationary process. (1p)

SZ15.) Let X_t and Y_t be from each other independent, stationary random processes with 0 expectation and f_X, f_Y spectral densities, furthermore let $Z_t = X_t \cdot Y_t$.

Show that the spectral density of process Z_t is $f_Z(x) = \int\limits_{-\pi}^{\pi} f_X(x-y) f_Y(y)\, dy$. (2p)

SZ16.) The spectral density function of a stationary process X_t is $f(x) = 100 \cdot I\left(-\frac{\pi}{6} - 0,01 < x < -\frac{\pi}{6} + 0,01\right) + 100 \cdot I\left(\frac{\pi}{6} - 0,01 < x < \frac{\pi}{6} + 0,01\right)$. Calculate $r(1) = \mathrm{cor}(X_t, X_{t+1})$! (2p)

SZ17.) The spectral density of an ARMA process is $f(x) = \frac{2+2\cos x}{\pi(1{,}5625-2{,}5\cos x+\cos^2 x)}$. Determine the order and the parameters of the process! (1p)