MULTIPLIER BOOTSTRAP IN GARCH MODELS

László Varga[1], András Zempléni[2]

Department of Probability Theory and Statistics, Institute of Mathematics, Faculty of Science, ELTE

[1]vargal4@cs.elte.hu, [2]zempleni@ludens.elte.hu

Abstract

Bootstrap methods are useful even if the data on hand are strongly dependent, especially in case of time series. Using the results of [1], we managed to prove that the multiplier (weighted) bootstrap quasi maximum likelihood estimation of the parameters of GARCH(p,q) processes is strongly consistent and the estimate's limit distribution is Gaussian. We assume that the weights are independent from the process, they are positive with probability one, there exists their first two moments and the correlation between them is weak. We examined the practical consequences of the theorem, the covariance matrix of the limit distribution and the rate of convergence via simulations.

Introduction - GARCH models

Definition 1 *$(X_t)_{t\in\mathbb{Z}}$ is called a GARCH(p,q) process if*

$$X_t = \sqrt{h_t}\eta_t \tag{1}$$

$$h_t = \omega_0 + \sum_{i=1}^{q}\alpha_{0i}X_{t-i}^2 + \sum_{j=1}^{p}\beta_{0j}h_{t-j} \tag{2}$$

where η_t $(t\in\mathbb{Z})$ are i.i.d. (0,1) random variables, $\omega_0 > 0, \alpha_{0i} \geq 0, \beta_{0j} \geq 0$ for $i = 1, ..., q$ and for $j = 1, ..., p$.

We denote the parameter vector by $\theta = (\theta_1, ..., \theta_{p+q+1})^T = (\omega, \alpha_1, ..., \alpha_q, \beta_1, ..., \beta_p)^T$, which belongs to the parameter space $\Theta = (0,\infty)\times[0,\infty)^{p+q}$.

The true value of the parameters, $\theta_0 = (\omega_0, \alpha_{01}, ..., \alpha_{0q}, \beta_{01}, ..., \beta_{0p})^T$ is unknown.

The following two theorems are fundamental in the theory of GARCH processes.

Theorem 1 *([1], page 37) If there exists a GARCH(p,q) process (1) - (2), which is second-order stationary, and if $\omega > 0$, then*

$$\sum_{i=1}^{q}\alpha_i + \sum_{j=1}^{p}\beta_j < 1.$$

If (1) holds, the unique strictly stationary solution of model (1) - (2) is a weak white noise.

The GARCH(p,q) process can be written in vector representation

$$\underline{z}_t = \underline{b}_t + A_t\underline{z}_{t-1} \qquad t\in\mathbb{Z}.$$

Theorem 2 *Let γ denote the top Ljapunov exponent of the matrix sequence $(A_t)_{t\in\mathbb{Z}}$.*
Then $\gamma < 0$ if and only if there exists a strictly stationary solution of the GARCH(p,q) model.

QML estimation of GARCH parameters

Assume that $\{x_1, \dots, x_n\}$ are observations from a GARCH(p,q) process (strictly stationary solution of the model). The Gaussian quasi-likelihood function, conditional on the $x_{1-q}, ..., x_0, \tilde\sigma_{1-p}^2, ..., \tilde\sigma_0^2$ initial values, is

$$L_n(\theta) = L_n(\theta; x_1, ..., x_n) = \prod_{i=1}^{n}\frac{1}{\sqrt{2\pi\tilde\sigma_t^2}}e^{-\frac{x_t^2}{2\tilde\sigma_t^2}}$$

where the $(\tilde\sigma_t^2)_{t\geq 1}$ are recursively defined with the following:

$$\tilde\sigma_t^2 = \tilde\sigma_t^2(\theta) = \omega + \sum_{i=1}^{q}\alpha_i x_{t-i}^2 + \sum_{j=1}^{p}\beta_j\tilde\sigma_{t-j}^2(\theta)$$

The QMLE of θ is defined as the solution $\hat\theta_n$ of

$$\hat\theta_n = \underset{\theta\in\Theta}{\text{argmax}}\ L_n(\theta) \tag{3}$$

To maximize the Gaussian likelihood function, we have to minimize the following function:

$$I_n(\theta) = \frac{1}{n}\sum_{t=1}^{n} l_t(\theta), \quad \text{where} \quad l_t(\theta) = \frac{x_t^2}{\tilde\sigma_t^2(\theta)} + \log(\tilde\sigma_t^2(\theta)).$$

Let $\mathcal{A}_\theta(z)$ and $\mathcal{B}_\theta(z)$ be the generating functions

$$\mathcal{A}_\theta(z) = \sum_{i=1}^{q}\alpha_i z^i \quad \text{and} \quad \mathcal{B}_\theta(z) = 1 - \sum_{j=1}^{p}\beta_j z^j$$

The following conditions are sufficient for the quasi-maximum likelihood estimator to have a normal limit distribution (see [1]).

Assumptions

A1. $\theta_0 \in \Theta$ and Θ is compact

A2. $\gamma(A_0) < 0$ and for all $\theta\in\Theta$, $\sum_{j=1}^{p}\beta_j < 1$

A3. η_t^2 has a nondegenerate distribution and $E\eta_t^2 = 1$

A4. If $p > 0$, $\mathcal{A}_{\theta_0}(z)$ and $\mathcal{B}_{\theta_0}(z)$ have no common roots, $\mathcal{A}_{\theta_0}(1) \neq 0, \alpha_{0q} + \beta_{0p} \neq 0$

A5. $\theta_0 \in \text{int}(\Theta)$

A6. $\kappa_\eta = E\eta_t^4 < \infty$

Theorem 3 *Let $(\hat\theta_n)_{n\geq 1}$ be a sequence of QMLEs satisfying (3), with initial conditions*

$$x_{1-q}^2 = ... = x_0^2 = x_1 \qquad \tilde\sigma_0^2 = ... = \tilde\sigma_{1-p}^2 = x_1^2 \tag{4}$$

*Under assumptions **A1-A4***

$$\hat\theta_n \xrightarrow[n\to\infty]{a.s.} \theta_0$$

Theorem 4 *Under assumptions **A1-A6***

$$\sqrt{n}(\hat\theta_n - \theta_0) \xrightarrow[n\to\infty]{d} N(0, (\kappa_\eta - 1)J^{-1})$$

where

$$J := E_{\theta_0}\left(\frac{\partial^2 l_t(\theta_0)}{\partial\theta\partial\theta^T}\right) = E_{\theta_0}\left(\frac{1}{\sigma_t^4(\theta_0)}\frac{\partial\sigma_t^2(\theta_0)}{\partial\theta}\frac{\partial\sigma_t^2(\theta_0)}{\partial\theta^T}\right) \tag{5}$$

With different assumptions, theorem 3 was first proved by [2] in 2003. Theorem 4 was proved by [2], and also by [3], who generalized the result to the case $E\eta_t^4 = \infty$.

Bootstrap QML estimation of GARCH parameters

We define the bootstrap weights as τ_{ni} $(1 \leq i \leq n,\ n \geq 1)$ triangular random variables independent from the process

$$\begin{matrix} \tau_{11} & & & \\ \tau_{21} & \tau_{22} & & \\ \vdots & \vdots & \ddots & \\ \tau_{n1} & \tau_{n2} & \dots & \tau_{nn} \\ \vdots & \vdots & & & \ddots \end{matrix}$$

Assumptions for the weights

B1. the weights are independent from the GARCH process

B2. $P(\tau_{ni} \geq 0) = 1 \quad 1 \leq i \leq n,\ n \geq 1$

B3. for all n, the first two moments of $\tau_{n1}, \dots, \tau_{nn}$ are finite and equal

B4. $\lim_{n\to\infty} E\tau_{ni} = 1 \quad i = 1, 2, ...$

B5. $\gamma := \lim_{n\to\infty} E\tau_{ni}^2 < \infty \quad i = 1, 2, ...$

B6. $r_n = R(\tau_{ni}, \tau_{nj}) \xrightarrow[n\to\infty]{} 0 \quad$ if $i \neq j$

The multinomial distribution is suitable choice for weights, it satisfies the three assumptions above.
Suitable choices for the weights:

$$(\tau_{n1}, ..., \tau_{nn}) \sim \text{ Multinom}\left(n; \frac{1}{n}, ..., \frac{1}{n}\right)$$

$$(\tau_{n1}, ..., \tau_{nn}) \sim \text{ i.i.d. Exp(1)}$$

$$(\tau_{n1}, ..., \tau_{nn}) \sim \text{ i.i.d. } \Gamma(n, n)$$

Now we modify the Gaussian likelihood function with the bootstrap weights, we have to minimize the following function:

$$I_n^*(\theta) = \frac{1}{n}\sum_{t=1}^{n} l_{nt}^*(\theta), \quad \text{where} \quad l_{nt}^*(\theta) = \tau_{nt}\left(\frac{x_t^2}{\tilde\sigma_t^2(\theta)} + \log(\tilde\sigma_t^2(\theta))\right)$$

For example if the weights are $(1, 2, 0, 1, ..., 1)$ then the second element of the sample is taken twice but the third one is missed.
The bootstrap QMLE of the parameter θ is defined as the solution $\hat\theta_n^*$ of

$$\hat\theta_n^* = \underset{\theta\in\Theta}{\text{argmax}}\ I_n^*(\theta) \tag{6}$$

Theorem 5 *Let $(\hat\theta_n^*)_{n\geq 1}$ be a sequence of bootstrap QMLEs satisfying (6), with initial conditions (4). Under assumptions **A1-A4** and **B1-B4***

$$\hat\theta_n^* \xrightarrow[n\to\infty]{a.s.} \theta_0$$

Theorem 6 *Under assumptions **A1-A6** and **B1-B6***

$$\sqrt{n}(\hat\theta_n^* - \theta_0) \xrightarrow[n\to\infty]{d} N\left(0, \gamma(\kappa_\eta - 1)J^{-1}\right) \tag{7}$$

where J is defined as in (5).

In our proof of theorems 5 and 6 we followed the methods of [1].

Simulations

The covariance matrix $(\kappa_\eta - 1)J^{-1}$ of the limit distribution depends on the true parameters. We analyzed this dependence in stationary ARCH(1) processes, where the parameters are $\omega_0 > 0$ and $0 < \alpha_0 < 1$.

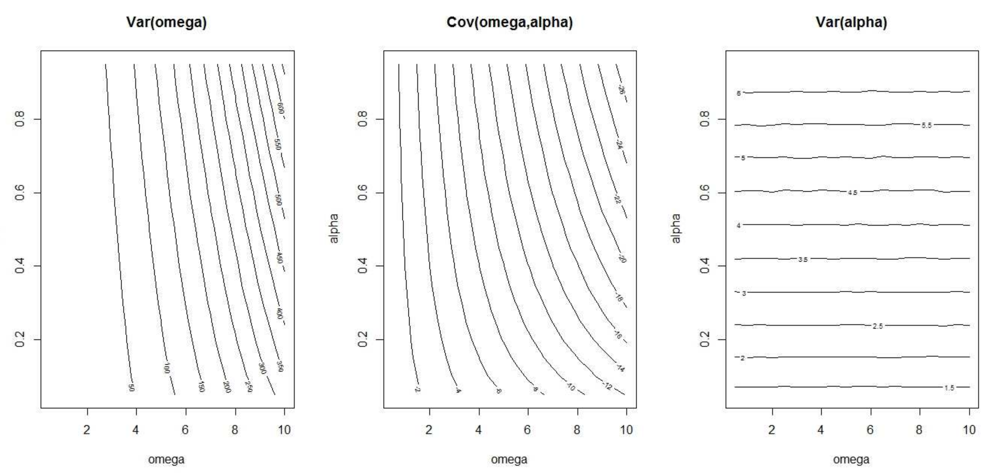

FIGURE 1: Contours of the elements of the limiting covariance matrix, ARCH(1) process

The variance of the estimated parameter $\hat\alpha$ does not seem to depend on the true parameter value omega. This is not trivial from the theoretical results and needs further investigation.
From now on we will concentrate on the ARCH(1) process with parameters $\omega_0 = 1$ and $\alpha_0 = 0.5$. Then the limiting covariance matrix of the QML estimation is

$$\begin{pmatrix} 4,893 & -2,148 \\ -2,148 & 3,926 \end{pmatrix}.$$

We drew 10^6 samples (with Gaussian innovations) of size 100 to 5000 and calculated the covariance matrix of the QML estimations.

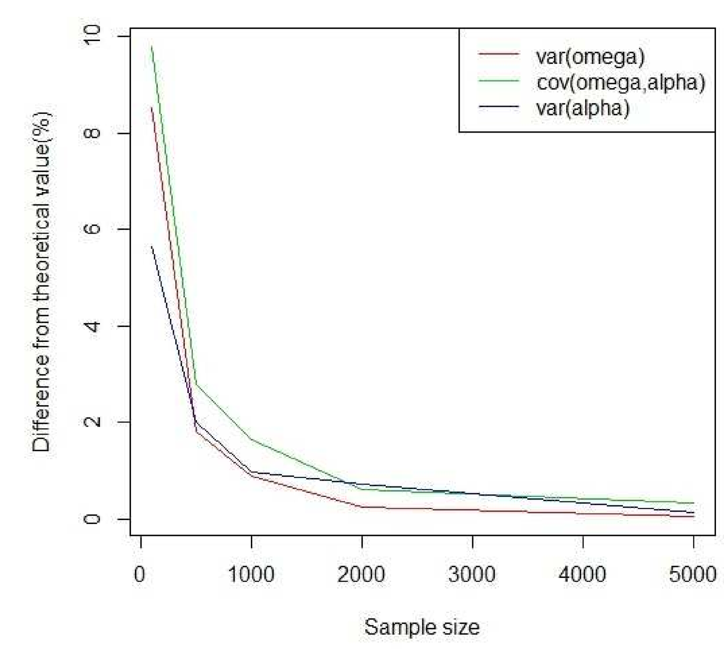

FIGURE 2: Convergence of the sample covariance matrix, ARCH(1) process, $\omega_0 = 1$ and $\alpha_0 = 0.5$

Figure 2 displays that the rate of convergence drastically improves until the sample size is under 1000 and just slightly after that.
The simulational experiences show that we must generate at least $5\cdot 10^5$ samples to properly estimate the covariance matrix, which takes a lot of time for the computer. Even the bootstrap can't help much if we draw too few samples as can be seen on Figure 3.

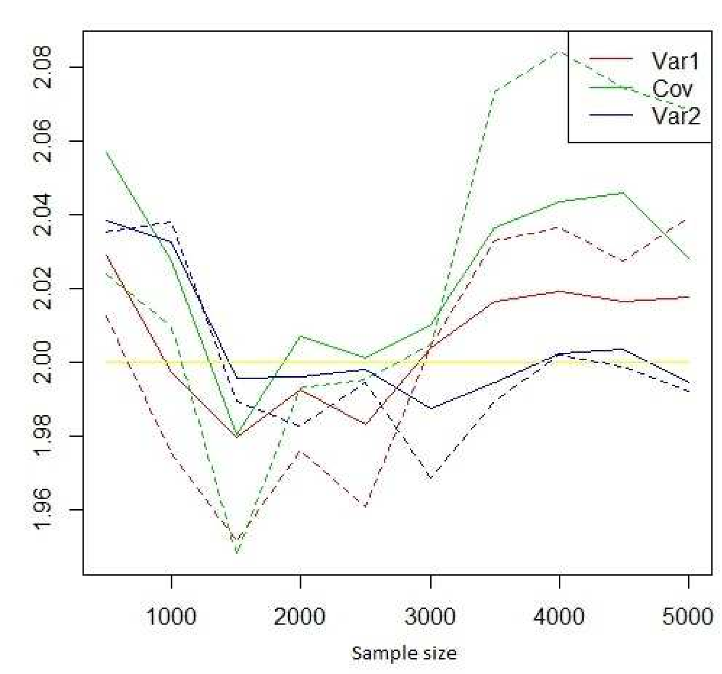

FIGURE 3: Convergence of the sample covariance matrix, ARCH(1) process, $\omega_0 = 1$ and $\alpha_0 = 0.5$, B=100 bootstrap replications

We drew 10^4 samples and these were bootstrapped 100 times with multinomial weights (so $\gamma = 2$). The dotted lines are the sample covariance matrix values without bootstrap weights, divided by the theoretical values and scaled to 2.

Acknowledgements

Our research was supported by TÁMOP-4.2.1/B-09/1/KMR-2010-0003 until the May of 2012.
My presentation was supported by Morgan Stanley.

References

[1] C. Francq, J-M. Zakoian (2010) GARCH models, Wiley

[2] I. Berkes, L. Horvath, P. Kokoszka (2003) GARCH processes: structure and estimation, Bernoulli, 9, pp. 201-227

[3] P. Hall, Q. Yao (2003) Inference in ARCH and GARCH models with heavy-tailed errors, Econometrica, 71, pp. 285-317